

# AN AXIOMATIC CHARACTERIZATION OF THE GABRIEL-ROITER MEASURE

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**ABSTRACT.** Given an abelian length category  $\mathcal{A}$ , the Gabriel-Roiter measure with respect to a length function  $\ell$  is characterized as a universal morphism  $\text{ind } \mathcal{A} \rightarrow P$  of partially ordered sets. The map is defined on the isomorphism classes of indecomposable objects of  $\mathcal{A}$  and is a suitable refinement of the length function  $\ell$ .

In his proof of the first Brauer-Thrall conjecture [5], Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories [1]. The first Brauer-Thrall conjecture asserts that every finite dimensional algebra of bounded representation type is of finite representation type. Ringel noticed (see the footnote on p. 91 of [1]) that the formalism of Gabriel and Roiter works equally well for studying the representations of algebras having unbounded representation type. We refer to recent work [2, 3, 4] for some beautiful applications.

In this note we present an axiomatic characterization of the Gabriel-Roiter measure which reveals its combinatorial nature. Given a finite dimensional algebra  $\Lambda$ , the Gabriel-Roiter measure is characterized as a universal morphism  $\text{ind } \Lambda \rightarrow P$  of partially ordered sets. The map is defined on the isomorphism classes of finite dimensional indecomposable  $\Lambda$ -modules and is a suitable refinement of the length function  $\text{ind } \Lambda \rightarrow \mathbb{N}$  which sends a module to its composition length.

The first part of this paper is purely combinatorial and might be of independent interest. We study length functions  $\lambda: S \rightarrow T$  on a fixed partially ordered set  $S$ . Such a length function takes its values in another partially ordered set  $T$ , for example  $T = \mathbb{N}$ . We denote by  $\text{Ch}(T)$  the set of finite chains in  $T$ , together with the lexicographic ordering. The map  $\lambda$  induces a new length function  $\lambda^*: S \rightarrow \text{Ch}(T)$ , which we call chain length function because each value  $\lambda^*(x)$  measures the lengths  $\lambda(x_i)$  of the elements  $x_i$  occurring in some finite chain  $x_1 < x_2 < \dots < x_n = x$  of  $x$  in  $S$ . We think of  $\lambda^*$  as a specific refinement of  $\lambda$  and provide an axiomatic characterization. It is interesting to observe that this construction can be iterated. Thus we may consider  $(\lambda^*)^*$ ,  $((\lambda^*)^*)^*$ , and so on.

The second part of the paper discusses the Gabriel-Roiter measure for a fixed abelian length category  $\mathcal{A}$ , for example the category of finite dimensional  $\Lambda$ -modules over some algebra  $\Lambda$ . For each length function  $\ell$  on  $\mathcal{A}$ , we consider its restriction to the partially ordered set  $\text{ind } \mathcal{A}$  of isomorphism classes of indecomposable objects of  $\mathcal{A}$ . Then the Gabriel-Roiter measure with respect to  $\ell$  is by definition the corresponding chain length function  $\ell^*$ . In particular, we obtain an axiomatic characterization of  $\ell^*$  and use it to reprove Gabriel's main property of the Gabriel-Roiter measure. Note that we work with a slight generalization of Gabriel's original definition. This enables us to characterize the injective objects of  $\mathcal{A}$  as those objects where  $\ell^*$  takes maximal values for some

length function  $\ell$ . This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category  $\mathcal{A}$ .

## 1. CHAINS AND LENGTH FUNCTIONS

**The lexicographic order on finite chains.** Let  $(S, \leq)$  be a partially ordered set. A subset  $X \subseteq S$  is a *chain* if  $x_1 \leq x_2$  or  $x_2 \leq x_1$  for each pair  $x_1, x_2 \in X$ . For a finite chain  $X$ , we denote by  $\min X$  its minimal and by  $\max X$  its maximal element, using the convention

$$\max \emptyset < x < \min \emptyset \quad \text{for all } x \in S.$$

We write  $\text{Ch}(S)$  for the set of all finite chains in  $S$  and let

$$\text{Ch}(S, x) := \{X \in \text{Ch}(S) \mid \max X = x\} \quad \text{for } x \in S.$$

On  $\text{Ch}(S)$  we consider the *lexicographic order* which is defined by

$$X \leq Y \quad :\Longleftrightarrow \quad \min(Y \setminus X) \leq \min(X \setminus Y) \quad \text{for } X, Y \in \text{Ch}(S).$$

**Remark 1.1.** (1)  $X \subseteq Y$  implies  $X \leq Y$  for  $X, Y \in \text{Ch}(S)$ .

(2) Suppose that  $S$  is totally ordered. Then  $\text{Ch}(S)$  is totally ordered. We may think of  $X \in \text{Ch}(S) \subseteq \{0, 1\}^S$  as a string of 0s and 1s which is indexed by the elements in  $S$ . The usual lexicographic order on such strings coincides with the lexicographic order on  $\text{Ch}(S)$ .

**Example 1.2.** Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mathbb{Q}$  be the set of rational numbers together with the natural ordering. Then the map

$$\text{Ch}(\mathbb{N}) \longrightarrow \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x}$$

is injective and order preserving, taking values in the interval  $[0, 1]$ . For instance, the subsets of  $\{1, 2, 3\}$  are ordered as follows:

$$\{\} < \{3\} < \{2\} < \{2, 3\} < \{1\} < \{1, 3\} < \{1, 2\} < \{1, 2, 3\}.$$

We need the following properties of the lexicographic order.

**Lemma 1.3.** Let  $X, Y \in \text{Ch}(S)$  and  $X^* := X \setminus \{\max X\}$ .

- (1)  $X^* = \max\{X' \in \text{Ch}(S) \mid X' < X \text{ and } \max X' < \max X\}$ .
- (2) If  $X^* < Y$  and  $\max X \geq \max Y$ , then  $X \leq Y$ .

*Proof.* (1) Let  $X' < X$  and  $\max X' < \max X$ . We show that  $X' \leq X^*$ . This is clear if  $X' \subseteq X^*$ . Otherwise, we have

$$\min(X^* \setminus X') = \min(X \setminus X') < \min(X' \setminus X) = \min(X' \setminus X^*),$$

and therefore  $X' \leq X^*$ .

- (2) The assumption  $X^* < Y$  implies by definition

$$\min(Y \setminus X^*) < \min(X^* \setminus Y).$$

We consider two cases. Suppose first that  $X^* \subseteq Y$ . If  $X \subseteq Y$ , then  $X \leq Y$ . Otherwise,

$$\min(Y \setminus X) < \max X = \min(X \setminus Y)$$

and therefore  $X < Y$ . Now suppose that  $X^* \not\subseteq Y$ . We use again that  $\max X \geq \max Y$ , exclude the case  $Y \subseteq X$ , and obtain

$$\min(Y \setminus X) = \min(Y \setminus X^*) < \min(X^* \setminus Y) = \min(X \setminus Y).$$

Thus  $X \leq Y$  and the proof is complete.  $\square$

**Length functions.** Let  $(S, \leq)$  be a partially ordered set. A *length function* on  $S$  is by definition a map  $\lambda: S \rightarrow T$  into a partially ordered set  $T$  satisfying for all  $x, y \in S$  the following:

- (L1)  $x < y$  implies  $\lambda(x) < \lambda(y)$ .
- (L2)  $\lambda(x) \leq \lambda(y)$  or  $\lambda(y) \leq \lambda(x)$ .
- (L3)  $\lambda_0(x) := \text{card}\{\lambda(x') \mid x' \in S \text{ and } x' \leq x\}$  is finite.

Two length functions  $\lambda$  and  $\lambda'$  on  $S$  are *equivalent* if

$$\lambda(x) \leq \lambda(y) \iff \lambda'(x) \leq \lambda'(y) \quad \text{for all } x, y \in S.$$

Observe that (L2) and (L3) are automatically satisfied if  $T = \mathbb{N}$ . A length function  $\lambda: S \rightarrow T$  induces for each  $x \in S$  a map

$$\text{Ch}(S, x) \longrightarrow \text{Ch}(T, \lambda(x)), \quad X \mapsto \lambda(X),$$

and therefore the following *chain length function*

$$S \longrightarrow \text{Ch}(T), \quad x \mapsto \lambda^*(x) := \max\{\lambda(X) \mid X \in \text{Ch}(S, x)\}.$$

Note that equivalent length functions induce equivalent chain length functions.

**Example 1.4.** (1) Let  $S$  be a poset such that for each  $x \in S$  there is a bound  $n_x \in \mathbb{N}$  with  $\text{card } X \leq n_x$  for all  $X \in \text{Ch}(S, x)$ . Then the map  $S \rightarrow \mathbb{N}$  sending  $x$  to  $\max\{\text{card } X \mid X \in \text{Ch}(S, x)\}$  is a length function.

(2) Let  $S$  be a poset such that  $\{x' \in S \mid x' \leq x\}$  is a finite chain for each  $x \in S$ . Then the map  $\lambda: S \rightarrow \mathbb{N}$  sending  $x$  to  $\text{card}\{x' \in S \mid x' \leq x\}$  is a length function. Moreover,  $\lambda^*$  is a length function and equivalent to  $\lambda$ .

(3) Let  $\lambda: S \rightarrow \mathbb{Z}$  be a length function which satisfies in addition the following properties of a *rank function*:  $\lambda(x) = \lambda(y)$  for each pair  $x, y$  of minimal elements of  $S$ , and  $\lambda(x) = \lambda(y) - 1$  whenever  $x$  is an immediate predecessor of  $y$  in  $S$ . Then  $\lambda^*$  is a length function and equivalent to  $\lambda$ .

**Basic properties.** Let  $\lambda: S \rightarrow T$  be a length function and  $\lambda^*: S \rightarrow \text{Ch}(T)$  the induced chain length function. We collect the basic properties of  $\lambda^*$ .

**Proposition 1.5.** *Let  $x, y \in S$ .*

- (C0)  $\lambda^*(x) = \max_{x' < x} \lambda^*(x') \cup \{\lambda(x)\}$ .
- (C1)  $x \leq y$  implies  $\lambda^*(x) \leq \lambda^*(y)$ .
- (C2)  $\lambda^*(x) = \lambda^*(y)$  implies  $\lambda(x) = \lambda(y)$ .
- (C3)  $\lambda^*(x') < \lambda^*(y)$  for all  $x' < x$  and  $\lambda(x) \geq \lambda(y)$  imply  $\lambda^*(x) \leq \lambda^*(y)$ .

The first property shows that the function  $\lambda^*: S \rightarrow \text{Ch}(T)$  can be defined by induction on the length  $\lambda_0(x)$  of the elements  $x \in S$ . The subsequent properties suggest to think of  $\lambda^*$  as a refinement of  $\lambda$ .

*Proof.* To prove (C0), let  $X = \lambda^*(x)$  and note that  $\max X = \lambda(x)$ . The assertion follows from Lemma 1.3 because we have

$$X \setminus \{\max X\} = \max\{X' \in \text{Ch}(T) \mid X' < X \text{ and } \max X' < \max X\}.$$

Now suppose  $x \leq y$  and let  $X \in \text{Ch}(S, x)$ . Then  $Y = X \cup \{y\} \in \text{Ch}(S, y)$  and we have  $\lambda(X) \leq \lambda(Y)$  since  $\lambda(X) \subseteq \lambda(Y)$ . Thus  $\lambda^*(x) \leq \lambda^*(y)$ . If  $\lambda^*(x) = \lambda^*(y)$ , then

$$\lambda(x) = \max \lambda^*(x) = \max \lambda^*(y) = \lambda(y).$$

To prove (C3), we use (C0) and apply Lemma 1.3 with  $X = \lambda^*(x)$  and  $Y = \lambda^*(y)$ . In fact,  $\lambda^*(x') < \lambda^*(y)$  for all  $x' < x$  implies  $X^* < Y$ , and  $\lambda(x) \geq \lambda(y)$  implies  $\max X \geq \max Y$ . Thus  $X \leq Y$ .  $\square$

**Corollary 1.6.** *Let  $\lambda: S \rightarrow T$  be a length function. Then the induced map  $\lambda^*$  is a length function.*

*Proof.* (L1) follows from (C1) and (C2). (L2) and (L3) follow from the corresponding conditions on  $\lambda$ .  $\square$

**An axiomatic characterization.** Let  $\lambda: S \rightarrow T$  be a length function. We present an axiomatic characterization of the induced chain length function  $\lambda^*$ . Thus we can replace the original definition in terms of chains by three simple conditions which express the fact that  $\lambda^*$  refines  $\lambda$ .

**Theorem 1.7.** *Let  $\lambda: S \rightarrow T$  be a length function. Then there exists a map  $\mu: S \rightarrow U$  into a partially ordered set  $U$  satisfying for all  $x, y \in S$  the following:*

- (M1)  $x \leq y$  implies  $\mu(x) \leq \mu(y)$ .
- (M2)  $\mu(x) = \mu(y)$  implies  $\lambda(x) = \lambda(y)$ .
- (M3)  $\mu(x') < \mu(y)$  for all  $x' < x$  and  $\lambda(x) \geq \lambda(y)$  imply  $\mu(x) \leq \mu(y)$ .

Moreover, for any map  $\mu': S \rightarrow U'$  into a partially ordered set  $U'$  satisfying the above conditions, we have

$$\mu'(x) \leq \mu'(y) \iff \mu(x) \leq \mu(y) \text{ for all } x, y \in S.$$

*Proof.* We have seen in Proposition 1.5 that  $\lambda^*$  satisfies (M1) – (M3). So it remains to show that for any map  $\mu: S \rightarrow U$  into a partially ordered set  $U$ , the conditions (M1) – (M3) uniquely determine the relation  $\mu(x) \leq \mu(y)$  for any pair  $x, y \in S$ . We proceed by induction on the length  $\lambda_0(x)$  of the elements  $x \in S$  and show in each step the following for  $S_n = \{x \in S \mid \lambda_0(x) \leq n\}$ .

- (i)  $\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\}$  is a finite set for all  $x \in S$ .
- (ii) (M1) – (M3) determine the relation  $\mu(x) \leq \mu(y)$  for all  $x, y \in S_n$ .
- (iii)  $\mu(x) \leq \mu(y)$  or  $\mu(y) \leq \mu(x)$  for all  $x, y \in S_n$ .

For  $n = 1$  the assertion is clear. In fact,  $S_1$  is the set of minimal elements in  $S$ , and  $\lambda(x) \geq \lambda(y)$  implies  $\mu(x) \leq \mu(y)$  for  $x, y \in S_1$ , by (M3). Now let  $n > 1$  and assume the assertion is true for  $S_{n-1}$ . To show (i), fix  $x \in S$ . The map

$$\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\} \longrightarrow \{\mu(x') \mid x' \in S_{n-1} \text{ and } x' \leq x\} \times \{\lambda(x') \mid x' \leq x\}$$

sending  $\mu(x')$  to the pair  $(\max_{y < x'} \mu(y), \lambda(x'))$  is well-defined by (i) and (iii); it is injective by (M3). Thus  $\{\mu(x') \mid x' \in S_n \text{ and } x' \leq x\}$  is a finite set. In order to verify (ii) and (iii), we choose for each  $x \in S_n$  a *Gabriel-Roiter filtration*, that is, a sequence

$$x_1 < x_2 < \dots < x_{\gamma(x)-1} < x_{\gamma(x)} = x$$

in  $S$  such that  $x_1$  is minimal and  $\max_{x' < x_i} \mu(x') = \mu(x_{i-1})$  for all  $1 < i \leq \gamma(x)$ . Such a filtration exists because the elements  $\mu(x')$  with  $x' < x$  form a finite chain, by (i) and (iii). Now fix  $x, y \in S_n$  and let  $I = \{i \geq 1 \mid \mu(x_i) = \mu(y_i)\}$ . We consider  $r = \max I$  and put  $r = 0$  if  $I = \emptyset$ . There are two possible cases. Suppose first that  $r = \gamma(x)$  or  $r = \gamma(y)$ . If  $r = \gamma(x)$ , then  $\mu(x) = \mu(x_r) = \mu(y_r) \leq \mu(y)$  by (M1). Now suppose  $\gamma(x) \neq r \neq \gamma(y)$ . Then we have  $\lambda(x_{r+1}) \neq \lambda(y_{r+1})$  by (M2) and (M3). If  $\lambda(x_{r+1}) > \lambda(y_{r+1})$ , then we obtain  $\mu(x_{r+1}) < \mu(y_{r+1})$ , again using (M2) and (M3). Iterating this argument, we get  $\mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{r+1})$ . From (M1) we get  $\mu(x) < \mu(y_{r+1}) \leq \mu(y)$ . Thus  $\mu(x) \leq \mu(y)$  or  $\mu(x) \geq \mu(y)$  and the proof is complete.  $\square$

**Corollary 1.8.** *Let  $\lambda: S \rightarrow T$  be a length function and let  $\mu: S \rightarrow U$  be a map into a partially ordered set  $U$  satisfying (M1) – (M3). Then  $\mu$  is a length function. Moreover, we have for all  $x, y \in S$*

$$\mu(x) = \mu(y) \iff \max_{x' < x} \mu(x') = \max_{y' < y} \mu(y') \text{ and } \lambda(x) = \lambda(y).$$

**Iterated length functions.** Let  $\lambda$  be a length function. Then  $\lambda^*$  is again a length function by Corollary 1.6. Thus we may define inductively  $\lambda^{(0)} = \lambda$  and  $\lambda^{(n)} = (\lambda^{(n-1)})^*$  for  $n \geq 1$ . In many examples, we have that  $\lambda^{(1)}$  and  $\lambda^{(3)}$  are equivalent. However, this is not a general fact. The author is grateful to Osamu Iyama for suggesting the following example.

**Example 1.9.** The following length functions  $\lambda^{(1)}$  and  $\lambda^{(3)}$  are not equivalent.

$$\begin{array}{ccc} \lambda^{(0)} : & \begin{array}{ccc} 4 & 5 & 6 \\ | \backslash & | \backslash & | \\ 3 & 2 & 1 \end{array} & \lambda^{(1)} : & \begin{array}{ccc} 3 & 6 & 5 \\ | \backslash & | \backslash & | \\ 1 & 2 & 4 \end{array} & \lambda^{(2)} : & \begin{array}{ccc} 6 & 4 & 2 \\ | \backslash & | \backslash & | \\ 5 & 3 & 1 \end{array} \\ \\ \lambda^{(3)} : & \begin{array}{ccc} 3 & 5 & 6 \\ | \backslash & | \backslash & | \\ 1 & 2 & 4 \end{array} & \lambda^{(4)} : & \begin{array}{ccc} 6 & 4 & 2 \\ | \backslash & | \backslash & | \\ 5 & 3 & 1 \end{array} \end{array}$$

## 2. ABELIAN LENGTH CATEGORIES

In this section we recall the definition and some basic facts about abelian length categories. We fix an abelian category  $\mathcal{A}$ .

**Subobjects.** We say that two monomorphisms  $\phi_1: X_1 \rightarrow X$  and  $\phi_2: X_2 \rightarrow X$  in  $\mathcal{A}$  are *equivalent*, if there exists an isomorphism  $\alpha: X_1 \rightarrow X_2$  such that  $\phi_1 = \phi_2 \circ \alpha$ . An equivalence class of monomorphisms into  $X$  is called a *subobject* of  $X$ . Given subobjects  $\phi_1: X_1 \rightarrow X$  and  $\phi_2: X_2 \rightarrow X$  of  $X$ , we write  $X_1 \subseteq X_2$  if there is a morphism  $\alpha: X_1 \rightarrow X_2$  such that  $\phi_2 = \phi_1 \circ \alpha$ . An object  $X \neq 0$  is *simple* if  $X' \subseteq X$  implies  $X' = 0$  or  $X' = X$ .

**Length categories.** An object  $X$  of  $\mathcal{A}$  has *finite length* if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \dots \subseteq X_{n-1} \subseteq X_n = X,$$

that is, each  $X_i/X_{i-1}$  is simple. Note that  $X$  has finite length if and only if  $X$  is both artinian (i.e. it satisfies the descending chain condition on subobjects) and noetherian (i.e. it satisfies the ascending chain condition on subobjects). An abelian category is

called a *length category* if all objects have finite length and if the isomorphism classes of objects form a set.

Recall that an object  $X \neq 0$  is *indecomposable* if  $X = X_1 \oplus X_2$  implies  $X_1 = 0$  or  $X_2 = 0$ . A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

**Example 2.1.** (1) The finitely generated modules over an artinian ring form a length category.

(2) Let  $k$  be a field and  $Q$  be any quiver. Then the finite dimensional  $k$ -linear representations of  $Q$  form a length category.

### 3. THE GABRIEL-ROITER MEASURE

Let  $\mathcal{A}$  be an abelian length category. The definition of the Gabriel-Roiter measure of  $\mathcal{A}$  is due to Gabriel [1] and was inspired by the work of Roiter [5]. We present a definition which is a slight generalization of Gabriel's original definition. Then we discuss some specific properties.

**Length functions.** A *length function* on  $\mathcal{A}$  is by definition a map  $\ell$  which sends each object  $X \in \mathcal{A}$  to some real number  $\ell(X) \geq 0$  such that

- ( $\ell 1$ )  $\ell(X) = 0$  if and only if  $X = 0$ , and
- ( $\ell 2$ )  $\ell(X) = \ell(X') + \ell(X'')$  for every exact sequence  $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ .

Note that such a length function is determined by the set of values  $\ell(S) > 0$ , where  $S$  runs through the isomorphism classes of simple objects of  $\mathcal{A}$ . This follows from the Jordan-Hölder Theorem. We write  $\ell_1$  for the length function satisfying  $\ell_1(S) = 1$  for every simple object  $S$ . Observe that  $\ell_1(X)$  is the usual composition length of an object  $X \in \mathcal{A}$ .

**The Gabriel-Roiter measure.** We consider the set  $\text{ind } \mathcal{A}$  of isomorphism classes of indecomposable objects of  $\mathcal{A}$  which is partially ordered via the subobject relation  $X \subseteq Y$ . Now fix a length function  $\ell$  on  $\mathcal{A}$ . The map  $\ell$  induces a length function  $\text{ind } \mathcal{A} \rightarrow \mathbb{R}$  satisfying (L1) – (L3), and the induced chain length function  $\ell^*: \text{ind } \mathcal{A} \rightarrow \text{Ch}(\mathbb{R})$  is by definition the *Gabriel-Roiter measure* of  $\mathcal{A}$  with respect to  $\ell$ . Gabriel's original definition [1] is based on the length function  $\ell_1$ . Whenever it is convenient, we substitute  $\mu = \ell^*$ .

**An axiomatic characterization.** The following axiomatic characterization of the Gabriel-Roiter measure is the main result of this note.

**Theorem 3.1.** *Let  $\mathcal{A}$  be an abelian length category and  $\ell$  a length function on  $\mathcal{A}$ . Then there exists a map  $\mu: \text{ind } \mathcal{A} \rightarrow P$  into a partially ordered set  $P$  satisfying for all  $X, Y \in \text{ind } \mathcal{A}$  the following:*

- (GR1)  $X \subseteq Y$  implies  $\mu(X) \leq \mu(Y)$ .
- (GR2)  $\mu(X) = \mu(Y)$  implies  $\ell(X) = \ell(Y)$ .
- (GR3)  $\mu(X') < \mu(Y)$  for all  $X' \subset X$  and  $\ell(X) \geq \ell(Y)$  imply  $\mu(X) \leq \mu(Y)$ .

Moreover, for any map  $\mu': \text{ind } \mathcal{A} \rightarrow P'$  into a partially ordered set  $P'$  satisfying the above conditions, we have

$$\mu'(X) \leq \mu'(Y) \iff \mu(X) \leq \mu(Y) \quad \text{for all } X, Y \in \text{ind } \mathcal{A}.$$

*Proof.* Use the axiomatic characterization of the chain length function  $\ell^*$  in Theorem 1.7.  $\square$

**Gabriel's main property.** Let  $\ell$  be a fixed length function on  $\mathcal{A}$ . The following main property of the Gabriel-Roiter measure  $\mu = \ell^*$  is crucial; it is the basis for all applications.

**Proposition 3.2** (Gabriel). *Let  $X, Y_1, \dots, Y_r \in \text{ind } \mathcal{A}$ . Suppose that  $X \subseteq Y = \bigoplus_{i=1}^r Y_i$ . Then  $\mu(X) \leq \max \mu(Y_i)$  and  $X$  is a direct summand of  $Y$  if  $\mu(X) = \max \mu(Y_i)$ .*

*Proof.* The proof only uses the properties (GR1) – (GR3) of  $\mu$ . Fix a monomorphism  $\phi: X \rightarrow Y$ . We proceed by induction on  $n = \ell_1(X) + \ell_1(Y)$ . If  $n = 2$ , then  $\phi$  is an isomorphism and the assertion is clear. Now suppose  $n > 2$ . We can assume that for each  $i$  the  $i$ th component  $\phi_i: X \rightarrow Y_i$  of  $\phi$  is an epimorphism. Otherwise choose for each  $i$  a decomposition  $Y'_i = \bigoplus_j Y_{ij}$  of the image of  $\phi_i$  into indecomposables. Then we use (GR1) and have  $\mu(X) \leq \max \mu(Y_{ij}) \leq \max \mu(Y_i)$  because  $\ell_1(X) + \ell_1(Y') < n$  and  $Y_{ij} \subseteq Y_i$  for all  $j$ . Now suppose that each  $\phi_i$  is an epimorphism. Thus  $\ell(X) \geq \ell(Y_i)$  for all  $i$ . Let  $X' \subset X$  be a proper indecomposable subobject. Then  $\mu(X') \leq \max \mu(Y_i)$  because  $\ell_1(X') + \ell_1(Y) < n$ , and  $X'$  is a direct summand if  $\mu(X') = \max \mu(Y_i)$ . We can exclude the case that  $\mu(X') = \max \mu(Y_i)$  because then  $X'$  is a proper direct summand of  $X$ , which is impossible. Now we apply (GR3) and obtain  $\mu(X) \leq \max \mu(Y_i)$ . Finally, suppose that  $\mu(X) = \max \mu(Y_i) = \mu(Y_k)$  for some  $k$ . We claim that we can choose  $k$  such that  $\phi_k$  is an epimorphism. Otherwise, replace all  $Y_i$  with  $\mu(X) = \mu(Y_i)$  by the image  $Y'_i = \bigoplus_j Y_{ij}$  of  $\phi_i$  as before. We obtain  $\mu(X) \leq \max \mu(Y_{ij}) < \mu(Y_k)$  since  $Y_{kj} \subset Y_k$  for all  $j$ , using (GR1) and (GR2). This is a contradiction. Thus  $\phi_k$  is an epimorphism and in fact an isomorphism because  $\ell(X) = \ell(Y_k)$  by (GR2). In particular,  $X$  is a direct summand of  $\bigoplus_i Y_i$ . This completes the proof.  $\square$

**Gabriel-Roiter filtrations.** We keep a length function  $\ell$  on  $\mathcal{A}$  and the corresponding Gabriel-Roiter measure  $\mu = \ell^*$ . Let  $X, Y \in \text{ind } \mathcal{A}$ . We say that  $X$  is a *Gabriel-Roiter predecessor* of  $Y$  if  $X \subset Y$  and  $\mu(X) = \max_{Y' \subset Y} \mu(Y')$ . Note that each object  $Y \in \text{ind } \mathcal{A}$  which is not simple admits a Gabriel-Roiter predecessor because  $\mu$  is a length function on  $\text{ind } \mathcal{A}$ . A Gabriel-Roiter predecessor  $X$  of  $Y$  is usually not unique, but the value  $\mu(X)$  is determined by  $\mu(Y)$ .

A sequence

$$X_1 \subset X_2 \subset \dots \subset X_{n-1} \subset X_n = X$$

in  $\text{ind } \mathcal{A}$  is called a *Gabriel-Roiter filtration* of  $X$  if  $X_1$  is simple and  $X_{i-1}$  is a Gabriel-Roiter predecessor of  $X_i$  for all  $1 < i \leq n$ . Clearly, each  $X$  admits such a filtration and the values  $\mu(X_i)$  are uniquely determined by  $X$ . Note that (C0) implies

$$(3.1) \quad \mu(X) = \{\ell(X_i) \mid 1 \leq i \leq n\}.$$

**Injective objects.** In order to illustrate Gabriel's main property, let us show that the Gabriel-Roiter measure detects injective objects. This is a remarkable fact because the Gabriel-Roiter measure is a combinatorial invariant, depending only on the poset of indecomposable objects and some length function, whereas the notion of injectivity involves all morphisms of the category  $\mathcal{A}$ .

**Theorem 3.3.** *An indecomposable object  $Q$  of  $\mathcal{A}$  is injective if and only if there is a length function  $\ell$  on  $\mathcal{A}$  such that  $\ell^*(X) \leq \ell^*(Q)$  for all  $X \in \text{ind } \mathcal{A}$ .*

We need the following lemma.

**Lemma 3.4.** *Let  $\ell$  be a length function on  $\mathcal{A}$  and fix indecomposable objects  $X, Y \in \mathcal{A}$ . Suppose that for each pair of simple subobjects  $X' \subseteq X$  and  $Y' \subseteq Y$ , we have  $\ell(X') < \ell(Y')$ . Then  $\ell^*(X) > \ell^*(Y)$ .*

*Proof.* We choose Gabriel-Roiter filtrations  $X_1 \subset \dots \subset X_n = X$  and  $Y_1 \subset \dots \subset Y_m = Y$ . Then  $\ell(X_1) < \ell(Y_1)$  and the formula (3.1) implies

$$\ell^*(X) = \{\ell(X_i) \mid 1 \leq i \leq n\} > \{\ell(Y_i) \mid 1 \leq i \leq m\} = \ell^*(Y).$$

□

*Proof of the theorem.* Suppose first that  $Q$  is injective. Then  $Q$  has a unique simple subobject  $S$  and we define a length function  $\ell = \ell_S$  on  $\mathcal{A}$  by specifying its values on each simple object  $T \in \mathcal{A}$  as follows:

$$\ell(T) := \begin{cases} 1 & \text{if } T \cong S, \\ 2 & \text{if } T \not\cong S. \end{cases}$$

Now let  $X \in \text{ind } \mathcal{A}$ . We claim that  $\ell^*(X) \leq \ell^*(Q)$ . To see this, let  $X' \subseteq X$  be the maximal subobject of  $X$  having composition factors isomorphic to  $S$ . Using induction on the composition length  $n = \ell_1(X')$  of  $X'$ , one obtains a monomorphism  $X' \rightarrow Q^n$ , and this extends to a map  $\phi: X \rightarrow Q^n$ , since  $Q$  is injective. Let  $X/X' = \oplus_i Y_i$  be a decomposition into indecomposables and  $\pi: X \rightarrow X/X'$  be the canonical map. Note that  $\ell^*(Y_i) < \ell^*(Q)$  for all  $i$  by our construction and Lemma 3.4. Then  $(\pi, \phi): X \rightarrow (\oplus_i Y_i) \oplus Q^n$  is a monomorphism and therefore  $\ell^*(X) \leq \ell^*(Q)$  by the main property.

Suppose now that  $\ell^*(X) \leq \ell^*(Q)$  for all  $X \in \text{ind } \mathcal{A}$  and some length function  $\ell$  on  $\mathcal{A}$ . To show that  $Q$  is injective, suppose that  $Q \subseteq Y$  is the subobject of some  $Y \in \mathcal{A}$ . Let  $Y = \oplus Y_i$  be a decomposition into indecomposables. Then the main property implies  $\ell^*(Q) \leq \max \ell^*(Y_i) \leq \ell^*(Q)$  and therefore  $Q$  is a direct summand of  $Y$ . Thus  $Q$  is injective and the proof is complete. □

Let us mention that there is the following analogous characterization of the simple objects of  $\mathcal{A}$ .

**Corollary 3.5.** *An indecomposable object  $S$  of  $\mathcal{A}$  is simple if and only if there is a length function  $\ell$  on  $\mathcal{A}$  such that  $\ell^*(S) \leq \ell^*(X)$  for all  $X \in \text{ind } \mathcal{A}$ .*

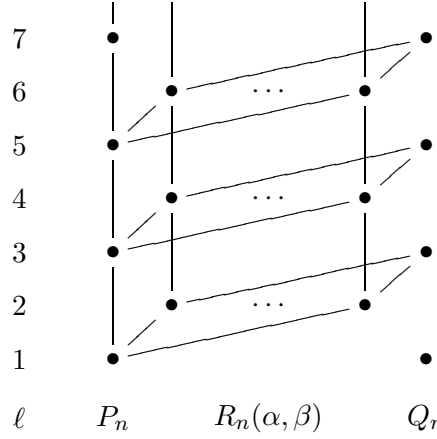
*Proof.* Use the property (GR1) of the Gabriel-Roiter measure and apply Lemma 3.4. □

**The Kronecker algebra.** Let  $\Lambda = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$  be the Kronecker algebra over an algebraically closed field  $k$ . We consider the abelian length category which is formed by all finite dimensional  $\Lambda$ -modules. A complete list of indecomposable objects is given by the preprojectives  $P_n$ , the regulars  $R_n(\alpha, \beta)$ , and the preinjectives  $Q_n$ . More precisely,

$$\text{ind } \Lambda = \{P_n \mid n \in \mathbb{N}\} \cup \{R_n(\alpha, \beta) \mid n \in \mathbb{N}, (\alpha, \beta) \in \mathbb{P}_k^1\} \cup \{Q_n \mid n \in \mathbb{N}\},$$



and we obtain the following Hasse diagram.



The set of indecomposables is ordered as follows via the Gabriel-Roiter measure with respect to  $\ell = \ell_1$ .

$$\ell^* : Q_1 = P_1 < P_2 < P_3 < \dots \quad R_1 < R_2 < R_3 < \dots \quad \dots < Q_4 < Q_3 < Q_2$$

$$(\ell^*)^* : Q_1 = P_1 < R_1 < Q_2 < P_2 < R_2 < Q_3 < P_3 < R_3 < Q_4 < \dots$$

Moreover,  $((\ell^*)^*)^*$  and  $\ell^*$  are equivalent length functions.

**Remark 3.6.** While  $\ell^*$  has been successfully employed for proving the first Brauer-Thrall conjecture, Hubery points out that  $(\ell^*)^*$  might be useful for proving the second. In fact, one needs to find a value  $(\ell^*)^*(X)$  such that the set  $\{X' \in \text{ind } \Lambda \mid (\ell^*)^*(X') = (\ell^*)^*(X)\}$  is infinite. The example of the Kronecker algebra shows that there exists such a value having only finitely many predecessors  $(\ell^*)^*(Y) < (\ell^*)^*(X)$ . Note that in all known examples  $((\ell^*)^*)^*$  and  $\ell^*$  are equivalent.

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